ULB-TH/05-07

Lovelock Terms and BRST Cohomology

Sandrine Cnockaert[a,1] and Marc Henneaux[a,b]

[a] Physique Théorique et Mathématique, Université Libre de Bruxelles & International Solvay Institutes,
U.L.B. Campus Plaine C.P. 231, B-1050, Bruxelles, Belgium

[b] Centro de Estudios Científicos, Casilla 1469, Valdivia, Chile

Abstract

Lovelock terms are polynomial scalar densities in the Riemann curvature tensor that have the remarkable property that their Euler-Lagrange derivatives contain derivatives of the metric of order not higher than two (while generic polynomial scalar densities lead to Euler-Lagrange derivatives with derivatives of the metric of order four). A characteristic feature of Lovelock terms is that their first nonvanishing term in the expansion $g_{\lambda\mu} = \eta_{\lambda\mu} + h_{\lambda\mu}$ of the metric around flat space is a total derivative. In this paper, we investigate generalized Lovelock terms defined as polynomial scalar densities in the Riemann curvature tensor *and* its covariant derivatives (of arbitrarily high but finite order) such that their first nonvanishing term in the expansion of the metric around flat space is a total derivative. This is done by reformulating the problem as a BRST cohomological one and by using cohomological tools. We determine all the generalized Lovelock terms. We find, in fact, that the class of nontrivial generalized Lovelock terms contains only the usual ones. Allowing covariant derivatives of the Riemann tensor does not lead to new structure. Our work provides a novel algebraic understanding of the Lovelock terms in the context of BRST cohomology.

[1] "Aspirant du F.N.R.S., Belgium"

1 Introduction

Lovelock terms are polynomial scalar densities in the Riemann curvature tensor (with indices saturated with $g^{\mu\nu}$)

$$\sqrt{-g}\, P(R_{\alpha\beta\gamma\delta}) \tag{1.1}$$

that have the remarkable property that their Euler-Lagrange derivatives contain derivatives of the metric of order not higher than two. By contrast, generic polynomial scalar densities lead to Euler-Lagrange derivatives with derivatives of the metric of order four. The most famous Lovelock term is probably the Einstein-Hilbert term itself,

$$a_{EH} = \sqrt{-g}\, R\,, \tag{1.2}$$

whose Euler-Lagrange derivatives yield the Einstein tensor. Lovelock terms have a long history [1] and have been systematically determined in all dimensions in [2, 3]. They have been considered as possible modifications of the Einstein-Hilbert Lagrangian in various contexts [4–6] and lead, in particular, to black hole solutions with interesting properties [6–9]. More recently, the quartic Lovelock term has been found to play an important role in deciphering the quantum correction structure of M-theory [10–14].

A characteristic feature of any Lovelock term a is that if one expands it according to the order of the field $h_{\mu\nu}$ and its derivatives,

$$a = a_k + a_{k+1} + \cdots \ , \tag{1.3}$$

then the first nonvanishing term a_k is a total derivative [5],

$$a_k = \partial_\mu V^\mu_k\,. \tag{1.4}$$

Here, each term in the expansion has definite polynomial order j,

$$N\, a_j = j\, a_j \tag{1.5}$$

where the operator counting the polynomial order is defined by

$$N = h_{\mu\nu}\frac{\partial}{\partial h_{\mu\nu}} + \partial_\rho h_{\mu\nu}\frac{\partial}{\partial(\partial_\rho h_{\mu\nu})} + \partial_\rho\partial_\sigma h_{\mu\nu}\frac{\partial}{\partial(\partial_\rho\partial_\sigma h_{\mu\nu})} + \cdots \tag{1.6}$$

The field $h_{\mu\nu}$ is the deviation of the metric $g_{\mu\nu}$ from the Minkowski metric $\eta_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{1.7}$$

That the property (1.4) must hold if a is a Lovelock term is easy to see since a_k is a polynomial in the linearized curvatures $K_{\alpha\beta\gamma\delta}$ (obtained by setting $g = -1$, $g_{\mu\nu} = \eta_{\mu\nu}$ in the bare metrics involved in (1.1) and by replacing $R_{\alpha\beta\gamma\delta}$ by $K_{\alpha\beta\gamma\delta}$) and thus reads

$$a_k = A^{\mu_1\mu_2\cdots\mu_{2k-1}\mu_{2k}\nu_1\nu_2\cdots\nu_{2k-1}\nu_{2k}}\ \partial^2_{\mu_1\mu_2} h_{\nu_1\nu_2} \cdots\ \partial^2_{\mu_{2k-1}\mu_{2k}} h_{\nu_{2k-1}\nu_{2k}} \tag{1.8}$$

for some constant coefficients $A^{\mu_1\mu_2\cdots\mu_{2k-1}\mu_{2k}\nu_1\nu_2\cdots\nu_{2k-1}\nu_{2k}}$. We recall that

$$K_{\alpha\beta\gamma\delta} = -K_{\beta\alpha\gamma\delta} = -K_{\alpha\beta\delta\gamma} = K_{\gamma\delta\alpha\beta} = \partial^2_{\delta[\alpha} h_{\beta]\gamma} - \partial^2_{\gamma[\alpha} h_{\beta]\delta}\,, \tag{1.9}$$

where brackets denote complete antisymmetrization with weight one. The Euler-Lagrange derivatives of a_k are

$$\frac{\delta a_k}{\delta h_{\rho\sigma}} = \partial^2_{\mu\nu}\left(\frac{\partial a_k}{\partial(\partial^2_{\mu\nu} h_{\rho\sigma})}\right) \tag{1.10}$$

and involve terms of order $k-1$ of the form $\partial^3 h\, \partial^3 h\, \partial^2 h \cdots \partial^2 h$, which are quadratic in the third derivatives of the metric, as well as terms of the form $\partial^4 h\, \partial^2 h \cdots \partial^2 h$, which are linear in the fourth derivatives. Being of polynomial order $k-1$, these terms cannot be cancelled by the contributions coming from the variational derivatives of the higher order terms a_j with $j > k$ since these contributions are of polynomial order $\geq k$. Hence, the Euler-Lagrange derivatives of a_k must identically vanish, which implies (1.4)[15–17].

Conversely, that the property (1.4) implies that a is a Lovelock term is a direct consequence of our analysis below. We can thus define the original Lovelock terms as the polynomial densities (1.1) in the curvature that have the central property (1.4).

In this paper, we investigate generalized Lovelock terms defined by adopting the property (1.4). More precisely, a (generalized) Lovelock term of order k is a polynomial scalar density in the Riemann tensor and its covariant derivatives of finite (but unrestricted) order,

$$a = \sqrt{-g}\, P(R_{\alpha\beta\gamma\delta}, D_\lambda R_{\alpha\beta\gamma\delta}, \cdots, D_{l_1} D_{\lambda_2} \ldots D_{\lambda_m} R_{\alpha\beta\gamma\delta}) \tag{1.11}$$

such that

- a starts at polynomial order k when expanded in the fields,

$$a = a_k + a_{k+1} + \cdots, \quad a_k \neq 0;$$

- the first term a_k is a total derivative, $a_k = \partial_\mu V^\mu_k$.

In (1.11), indices are contracted with the inverse $g^{\alpha\beta}$ of the spacetime metric while D_λ denotes the covariant derivative. The property $a_k = \partial_\mu V^\mu$ is necessary for the derivatives of the metric of highest expected order to drop out from the Euler-Lagrange derivatives – and, as we shall see, it turns out to be also sufficient.

It is clear from our definition that if a is a Lovelock term of order k, then $a + b^{(k+1)} + D_\mu T^\mu$, where $b^{(k+1)}$ starts at polynomial order $k+1$ and T^μ is a vector density, is also a Lovelock term of order k (even if $b^{(k+1)}$ is not a Lovelock term of order $k+1$). We shall consider two such Lovelock terms of order k as being equivalent. Starting from the Lovelock terms of order 1, one can successively construct the Lovelock terms of increasing orders 2, 3, etc.

Our main result is that there are, in fact, no new Lovelock terms besides those already derived in [2, 3], even if one allows, as here, derivatives of the Riemann tensor. Accordingly, nontrivial Lovelock terms of order k can be assumed to be polynomials of order k in the undifferentiated Riemann tensor and are proportional to

$$\sqrt{-g}\,\delta^{\mu_1\mu_2\cdots\mu_{2k}}_{\nu_1\nu_2\cdots\nu_{2k}} R^{\nu_1\nu_2}{}_{\mu_1\mu_2}\cdots R^{\nu_{2k-1}\nu_{2k}}{}_{\mu_{2k-1}\mu_{2k}} \tag{1.12}$$

with

$$\delta^{\mu_1\mu_2\cdots\mu_{2k}}_{\nu_1\nu_2\cdots\nu_{2k}} = \delta^{[\mu_1}_{[\nu_1}\delta^{\mu_2}_{\nu_2}\cdots\delta^{\mu_{2k}]}_{\nu_{2k}]}\,.$$

Note that while the polynomial order N is not homogeneous, one may assume that the derivative order K defined by

$$K\,P = \sum_s s\,\partial_{\rho_1\cdots\rho_s}h_{\mu\nu}\frac{\partial P}{\partial(\partial_{\rho_1\cdots\rho_s}h_{\mu\nu})}, \tag{1.13}$$

is homogeneous since the Riemann tensor is homogeneous of derivative order 2 and the Euler-Lagrange operator preserves the derivative order. For a given even derivative order $s = 2k$, there is only one nontrivial Lovelock term of order k, namely (1.12), while there is no nontrivial Lovelock term of odd derivative order.

Our approach relies on the formulation of the problem in terms of BRST cohomology. We show that Lovelock terms define cohomological classes of $H(\gamma_0|d)$ of form degree n and ghost number zero, which are Lorentz-invariant, while non-Lovelock terms define cohomological classes of $H(\gamma_0)$. Here, n is

the spacetime dimension, γ_0 is the "longitudinal differential along the gauge orbits" of the linearized theory acting on the fields and the ghosts only (and their derivatives but not on the antifields),

$$\gamma_0 h_{\mu\nu} = \partial_\mu C_\nu + \partial_\nu C_\mu, \quad \gamma_0 C_\mu = 0 \tag{1.14}$$

(which plays a crucial role in BRST theory [18, 19]) while d is the spacetime exterior differential. In (1.14), C^μ are the diffeomorphism ghosts and their index is lowered with the flat metric,

$$C_\mu = \eta_{\mu\nu} C^\nu. \tag{1.15}$$

Standard techniques of homological algebra as well as known results on the BRST cohomology for gravity then enable one to completely determine all the generalized Lovelock terms. In fact, once the problem is reformulated cohomologically, the determination of the Lovelock terms is quite immediate (section 3).

We recall that the equivalence classes $[m]$ of $H(\gamma_0|d)$ are defined by the cocycle condition

$$\gamma_0 m + dq = 0 \tag{1.16}$$

(for some q), with

$$m \sim m' \text{ iff } m - m' = \gamma_0 p + dr \tag{1.17}$$

for some p and r. As usual, we switch between form notations ($\gamma_0 m + dq = 0$) and their duals ($\gamma_0 m + \partial_\mu q^\mu = 0$ for a n-form, $\gamma_0 m^\mu + \partial_\nu q^{\mu\nu} = 0$ with $q^{\mu\nu} = -q^{\nu\mu}$ for a $(n-1)$-form etc).

Our paper is organized as follows. In the next section, we reformulate the problem as a cohomological problem in terms of the BRST differential of the free theory. We then solve this cohomological problem completely, determining thereby all nontrivial Lovelock terms (section 3). To that effect, we use some cohomological results on $H(\gamma_0|d)$ established in the appendices. We finally comment on our results (sections 4 and 5).

2 Formulating the problem as a cohomological problem

Let a be a polynomial density in the curvature and its covariant derivatives, as in (1.11). One has

$$\gamma a = \partial_\mu (C^\mu a). \tag{2.1}$$

Here γ is the longitudinal differential along the gauge orbits of the full Einstein theory,

$$\gamma g_{\mu\nu} = \mathcal{L}_C g_{\mu\nu}, \quad \gamma C^\mu = C^\rho \partial_\rho C^\mu \tag{2.2}$$

where $\mathcal{L}_C$ is the Lie derivative along C^μ. If one expands (2.1) according to the polynomial degree, one gets as first two equations

$$\gamma_0 a_k = 0\,, \tag{2.3}$$

$$\gamma_1 a_k + \gamma_0 a_{k+1} = \partial_\mu (C^\mu a_k)\,, \tag{2.4}$$

where $\gamma = \gamma_0 + \gamma_1 + \cdots$ is the expansion of γ according to the polynomial degree. We assume that $a_k \neq 0$. Thus a_k is a nonvanishing polynomial in the linearized curvature $K_{\alpha\beta\gamma\delta}$ and its ordinary derivatives; it contains therefore at least $2k$ derivative operators. As explained in the introduction, we can assume that each term has definite derivative order t, with $t \geq 2k$,

$$N(a_j) = j\, a_j, \tag{2.5}$$

$$K(a_j) = t\, a_j. \tag{2.6}$$

From now on, we shall call γ_0 the BRST differential (even though it is only a piece of it, but the full BRST differential will not be encountered in this paper any more).

Now, let us assume that a is a Lovelock term of order k. Then

$$a_k = \partial_\mu V^\mu_k \tag{2.7}$$

where $K(V^\mu_k) = (t-1)V^\mu_k$. The equations (2.3) and (2.4) become

$$\partial_\mu(\gamma_0 V^\mu_k) = 0\ , \tag{2.8}$$

$$\gamma_0 a_{k+1} = \partial_\mu T^\mu_{k+1} \tag{2.9}$$

with

$$T^\mu_{k+1} = C^\mu \partial_\rho V^\rho_k - \gamma_1 V^\mu_k. \tag{2.10}$$

It follows from (2.8) and the triviality of d that

$$\gamma_0 V^\mu_k = \partial_\nu V^{\mu\nu}_{k|1} \tag{2.11}$$

for some $V^{\mu\nu}_{k|1} = -V^{\nu\mu}_{k|1}$ of ghost number one. Thus, we see that V^μ_k and a_{k+1} both fulfill the cocycle condition of the γ_0-cohomology modulo d, the

former in form degree $n-1$ and the latter in form degree n. Their respective polynomial, derivative and ghost degrees are

$$N(V_k^\mu) = k\, V_k^\mu,\ K(V_k^\mu) = (t-1)\, V_k^\mu,\ gh(V_k^\mu) = 0 \tag{2.12}$$
$$N(a_{k+1}) = (k+1)\, a_{k+1},\ K(a_{k+1}) = t\, a_{k+1},\ gh(a_{k+1}) = 0. \tag{2.13}$$

Furthermore, V_k^μ cannot be trivial ($V_k^\mu = \partial_\nu S^{\mu\nu}$ for some $S^{\mu\nu} = -S^{\nu\mu}$) since otherwise a_k would be zero. Thus V_k^μ defines a nontrivial cohomological class of $H^{n-1,0}_{k,t-1}(\gamma_0|d)$, where in $H^{i,j}_{k,l}$, the suffices i,j and the indices k,l are respectively the form degree, the ghost number, the polynomial degree and the derivative degree. For expressions involving the ghosts, the polynomial and derivative degrees are respectively extended as

$$N = \sum_{s\geq 0} \left(\partial_{\rho_1\cdots\rho_s} h_{\mu\nu} \frac{\partial}{\partial(\partial_{\rho_1\cdots\rho_s} h_{\mu\nu})} + \partial_{\rho_1\cdots\rho_s} C_\mu \frac{\partial}{\partial(\partial_{\rho_1\cdots\rho_s} C_\mu)} \right), \tag{2.14}$$
$$K = \sum_{s\geq 1} s \left(\partial_{\rho_1\cdots\rho_s} h_{\mu\nu} \frac{\partial}{\partial(\partial_{\rho_1\cdots\rho_s} h_{\mu\nu})} + \partial_{\rho_1\cdots\rho_s} C_\mu \frac{\partial}{\partial(\partial_{\rho_1\cdots\rho_s} C_\mu)} \right). \tag{2.15}$$

We shall see below that, similarly, a_{k+1} defines a nontrivial element of $H^{n,0}_{k+1,t}(\gamma_0|d)$.

3 Determining V_k^μ

Our analysis is based on the resolution of the equation (2.11) for V_k^μ. For that purpose, we use the standard descent techniques, which rely on the triviality of the cohomology of d in form degree $< n$ (and $\neq 0$). One gets from (2.11) the chain of equations

$$\gamma_0 V_k^\mu = \partial_\nu V_{k|1}^{\mu\nu}, \tag{3.1}$$
$$\gamma_0 V_{k|1}^{\mu\nu} = \partial_\rho V_{k|2}^{\mu\nu\rho} \tag{3.2}$$
$$\vdots$$
$$\gamma_0 V_{k|s-1}^{\mu_1\cdots\mu_s} = \partial_{\mu_{s+1}} V_{k|s}^{\mu_1\cdots\mu_s\mu_{s+1}} \tag{3.3}$$
$$\gamma_0 V_{k|s}^{\mu_1\cdots\mu_s\mu_{s+1}} = 0 \tag{3.4}$$

where all the $V_{k|j-1}^{\mu_1\cdots\mu_j}$ are totally antisymmetric in their upper indices. The descent stops at some ghost number s (and dual form degree $n-s-1$) since there is no p-form with $p<0$. The antisymmetric tensors have all polynomial

degree k and derivative order $t-1 \geq 2k-1$. If the last term $V^{\mu_1\cdots\mu_s\mu_{s+1}}_{k|s}$ is trivial in $H^{n-s-1,s}_{k,t-1}$,

$$V^{\mu_1\cdots\mu_s\mu_{s+1}}_{k|s} = \gamma_0 M^{\mu_1\cdots\mu_s\mu_{s+1}} + \partial_{\mu_{s+2}} M^{\mu_1\cdots\mu_s\mu_{s+1}\mu_{s+2}}, \tag{3.5}$$

then one can remove $V^{\mu_1\cdots\mu_{s+1}}_{k|s}$ by redefinitions that affect $V^{\mu_1\cdots\mu_s}_{k|s-1}$ as $V^{\mu_1\cdots\mu_s}_{k|s-1} \rightarrow V'^{\mu_1\cdots\mu_s}_{k|s-1} = V^{\mu_1\cdots\mu_s}_{k|s-1} - \partial_{\mu_{s+1}} M^{\mu_1\cdots\mu_s\mu_{s+1}}$ and do not affect the preceding V's, and assume the chain stops one step earlier, at $V'^{\mu_1\cdots\mu_s}_{k|s-1}$,

$$\gamma_0 V'^{\mu_1\cdots\mu_s}_{k|s-1} = 0. \tag{3.6}$$

In order to analyse further the descent, we need to consider two cases: (i) $t > 2k$, and (ii) $t = 2k$.

3.1 Case $t > 2k$

The analysis of Appendix B shows that if a γ_0-cocycle (i.e., a solution of (3.4)) can be lifted as in (3.3) then it can be expressed as a polynomial in the linearized curvature two-form with coefficients that involve the undifferentiated ghosts C_μ and their antisymmetrized derivatives

$$H_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu, \tag{3.7}$$

up to irrelevant trivial terms. Derivatives of the linearized curvature appear only in the trivial terms. Now, $V^{\mu_1\cdots\mu_s\mu_{s+1}}_{k|s}$ involves s ghosts and thus only $k-s$ linearized curvatures. These $k-s$ curvatures take up $2k-2s$ derivatives, leaving $t-1-(2k-2s) \geq 2s$ derivatives for the s ghosts. But s ghosts can only take at most $s < 2s$ ($s > 0$) derivatives in a nontrivial term, which implies that $V^{\mu_1\cdots\mu_s\mu_{s+1}}_{k|s}$ is trivial for $s \geq 1$. Thus, we can assume that the descent stops immediately at $\gamma_0 V^\mu_k = 0$, i.e., that V^μ_k is invariant for the linearized gauge transformations.

If V^μ_k is gauge invariant, it is a function of the linearized curvature and its derivatives[2],

$$V^\mu_k = V^\mu_k([K_{\alpha\beta\gamma\delta}]) \tag{3.8}$$

[2]In this equation and in the equations below, we use the standard notation $f = f([\phi])$ for a function f of the field ϕ and a finite number of its derivatives.

The indices in V_k^μ are contracted with the Lorentz metric $\eta^{\mu\nu}$ so that V_k^μ is a Lorentz vector (a_k is a Lorentz scalar). Let $\mathcal{V}^\mu$ be the vector density

$$\mathcal{V}^\mu = \sqrt{-g}\, V^\mu \tag{3.9}$$

where V^μ is obtained from V_k^μ by replacing $\eta^{\mu\nu}$ by $g^{\mu\nu}$ and the linearized curvatures $K_{\alpha\beta\gamma\delta}$ and their derivatives by the Riemann tensor $R_{\alpha\beta\gamma\delta}$ and their covariant derivatives. It is clear that $a - D_\mu \mathcal{V}^\mu$ starts with a term of order $k+1$ since $(D_\mu \mathcal{V}^\mu)_k = \partial_\mu V_k^\mu = a_k$. Accordingly, there is no nontrivial generalized Lovelock term of order k if the derivative order t is strictly greater than $2k$.

3.2 Case $t = 2k$

When $t = 2k$, the analysis proceeds as above, but the descent can be shortened only to two steps,

$$\gamma_0 V_k^\mu = \partial_\nu V_{k|1}^{\mu\nu}, \tag{3.10}$$

$$\gamma_0 V_{k|1}^{\mu\nu} = 0 \tag{3.11}$$

The antisymmetric tensor $V_{k|1}^{\mu\nu}$ contains $k-1$ undifferentiated curvatures (which take up $2k-2$ derivatives) and one $H_{\mu\nu}$ (which takes up the remaining derivative). As explained in the appendix B, it takes the form

$$V_{k|1}^{\mu\nu} = B^{[\mu\nu\sigma_1\ldots\sigma_{2k-2}]}_{\beta_1\beta_2\rho_1\ldots\rho_{2k-2}} K_{\sigma_1\sigma_2}{}^{\rho_1\rho_2} \ldots K_{\sigma_{2k-3}\sigma_{2k-2}}{}^{\rho_{2k-3}\rho_{2k-2}} H^{\beta_1\beta_2} \tag{3.12}$$

(see (B.5)). Lorentz invariance forces the constant tensor $B^{[\mu\nu\sigma_1\ldots\sigma_{2k-2}]}_{\beta_1\beta_2\rho_1\ldots\rho_{2k-2}}$ to be proportional to an antisymmetric product of Kronecker δ's,

$$B^{[\mu\nu\sigma_1\ldots\sigma_{2k-2}]}_{\beta_1\beta_2\rho_1\ldots\rho_{2k-2}} = \alpha\, \delta^{\mu\nu\sigma_1\ldots\sigma_{2k-2}}_{\beta_1\beta_2\rho_1\ldots\rho_{2k-2}} \tag{3.13}$$

where α is some constant. [If $n = 2k$, there is another possibility proportional to $\epsilon^{\mu\nu\sigma_1\cdots\sigma_{n-2}}$, but it leads to a solution a which is a total derivative (not just a_k, but the whole a is a total derivative - in fact the Pontryagin class). The term (3.12) with B given by (3.13) yields in fact also a total derivative when $n = 2k$.]

Thus, $V_{k|1}^{\mu\nu}$ reads (up to trivial terms)

$$V_{k|1}^{\mu\nu} = \alpha\, \delta^{\mu\nu\sigma_1\ldots\sigma_{2k-2}}_{\beta_1\beta_2\rho_1\ldots\rho_{2k-2}} K_{\sigma_1\sigma_2}{}^{\rho_1\rho_2} \ldots K_{\sigma_{2k-3}\sigma_{2k-2}}{}^{\rho_{2k-3}\rho_{2k-2}} H^{\beta_1\beta_2}\,. \tag{3.14}$$

It follows that

$$V_k^\mu = 2\,\alpha\,\delta^{\mu\nu\sigma_1\dots\sigma_{2k-2}}_{\beta_1\beta_2\rho_1\dots\rho_{2k-2}} K_{\sigma_1\sigma_2}{}^{\rho_1\rho_2}\dots K_{\sigma_{2k-3}\sigma_{2k-2}}{}^{\rho_{2k-3}\rho_{2k-2}}\partial^{\beta_1}h^{\beta_2}{}_\nu \tag{3.15}$$

up to strictly invariant terms. These invariant terms can be removed as above so that we can indeed assume that V_k^μ is given by (3.15). Computing the divergence of V_k^μ yields then

$$a_k = -\alpha\,\delta^{\sigma_1\dots\sigma_{2k}}_{\rho_1\dots\rho_{2k}} K_{\sigma_1\sigma_2}{}^{\rho_1\rho_2}\dots K_{\sigma_{2k-1}\sigma_{2k}}{}^{\rho_{2k-1}\rho_{2k}}\,, \tag{3.16}$$

which is the linearization of the standard Lovelock term of order k. Covariantizing leads to

$$a = -\alpha\,\sqrt{-g}\;\delta^{\sigma_1\dots\sigma_{2k}}_{\rho_1\dots\rho_{2k}} R_{\sigma_1\sigma_2}{}^{\rho_1\rho_2}\dots R_{\sigma_{2k-1}\sigma_{2k}}{}^{\rho_{2k-1}\rho_{2k}} \tag{3.17}$$

up to terms of order $> k$. This term is the complete Lovelock term of order k [2, 3]. It follows that the standard Lovelock terms exhaust all the possible Lovelock terms.

4 Comments on a_{k+1}

We have found all the Lovelock terms by solving Eq. (2.11), which expresses that V_k^μ is a cocycle of $H(\gamma_0|d)$. One can alternatively focus on Eq.(2.9) and determine a_{k+1}, which is also a cocycle of $H(\gamma_0|d)$. The interest of a_{k+1} is that it coincides with the Pauli-Fierz Lagrangian for the Lovelock term of order one (in which case a_1 is a total derivative and $a_2 = \mathcal{L}_{PF}$) and provides generalizations of the Pauli-Fierz Lagrangian for higher k's.

We shall only sketch here how the analysis proceeds, without giving all the details. Nontrivial a_{k+1}'s exist only if $t = 2k$ as can be seen by examining the descent associated with a_{k+1}, namely, $\gamma_0 a_{k+1} = \partial_\mu T^\mu_{k+1}$, $\gamma_0 T^\mu_{k+1} = \partial_\nu T^{\mu\nu}_{k+1}$, etc, and using derivative counting arguments similar to the ones used above. Furthermore, one finds that when $t = 2k$, the descent must stop after one step,

$$\gamma_0 a_{k+1} = \partial_\mu T^\mu_{k+1}, \tag{4.1}$$

$$\gamma_0 T^\mu_{k+1} = 0. \tag{4.2}$$

The term T^μ_{k+1} has ghost number one and contains k curvatures; hence it can only involve the undifferentiated ghost C_μ. Using Lorentz invariance, one gets

$$T^\mu_{k+1} = \beta\,\delta^{\mu\sigma_1\dots\sigma_{2k}}_{\nu\rho_1\dots\rho_{2k}} K_{\sigma_1\sigma_2}{}^{\rho_1\rho_2}\dots K_{\sigma_{2k-1}\sigma_{2k}}{}^{\rho_{2k-1}\rho_{2k}} C^\nu\,, \tag{4.3}$$

which yields

$$a_{k+1} = \beta \, \frac{1}{2} \delta^{\mu\sigma_1 \ldots \sigma_{2k}}_{\nu\rho_1 \ldots \rho_{2k}} K_{\sigma_1\sigma_2}{}^{\rho_1\rho_2} \ldots K_{\sigma_{2k-1}\sigma_{2k}}{}^{\rho_{2k-1}\rho_{2k}} h_\mu^{\ \nu} \,. \tag{4.4}$$

One may rewrite a_{k+1} in the more suggestive form

$$a_{k+1} = \beta \, \mathcal{G}_k^{\mu\nu} \, h_{\mu\nu} \tag{4.5}$$

where the tensor $\mathcal{G}_k^{\mu\nu}$ is given by

$$\mathcal{G}_k^{\mu\nu} = \frac{1}{2} \, \delta^{\mu\sigma_1 \ldots \sigma_{2k}}_{\nu\rho_1 \ldots \rho_{2k}} K_{\sigma_1\sigma_2}{}^{\rho_1\rho_2} \ldots K_{\sigma_{2k-1}\sigma_{2k}}{}^{\rho_{2k-1}\rho_{2k}} \tag{4.6}$$

and fulfills

$$\mathcal{G}_k^{\mu\nu} = \mathcal{G}_k^{\nu\mu}, \qquad \partial_\mu \mathcal{G}_k^{\mu\nu} = 0 \tag{4.7}$$

(identically).

Note also that

$$\frac{\delta a_{k+1}}{\delta h_{\alpha\beta}} = \beta(k+1) \, \mathcal{G}_k^{\alpha\beta} \tag{4.8}$$

so that

$$\frac{\delta a}{\delta g_{\alpha\beta}} = -\frac{\alpha}{2} \sqrt{-g} \, \Xi^{\alpha\beta} \tag{4.9}$$

where $\Xi^{\alpha\beta}$ is the covariantization of $\mathcal{G}_k^{\alpha\beta}$. One must take $\beta = -\frac{\alpha}{2\,(k+1)}$ to match the normalization of a adopted in the preceding section.

5 Conclusions

In this paper, we have studied the Lovelock terms of order k, defined as polynomial densities in the curvature tensor and its covariant derivatives such that the first term in an expansion around flat space is a total derivative. We have found that Lovelock terms may be assumed to involve only the undifferentiated Riemann tensor and are thus exhausted by the standard Lovelock solutions [2, 3]. Allowing covariant derivatives of the curvature tensor does not lead to new solutions. This is perhaps not too surprising in view of the topological interpretation of the Lovelock terms, which can be viewed as the dimensional continuation of characteristic classes [5].

Our approach is based on a BRST cohomological reformulation of the problem and provides a new light on the significance of the Lovelock terms.

In the full theory, there is not much difference between Lovelock terms and generic polynomials in the curvature from the invariance point of view, since all define nontrivial elements of $H(\gamma|d)$. However, when restricted to the linear theory, there is a clear difference between the two: the former define nontrivial elements of $H(\gamma_0|d)$, while the latter define nontrivial elements of $H(\gamma_0)$. In that respect, it is of interest to point out that the cohomological groups $H(\gamma_0|d)$ for linearized gravity have not been computed yet in all ghost numbers and ghost degrees. [These groups are known for p-forms [16, 20], but not for more general tensor fields. They are important for determining Lagrangians for higher spin gauge fields, see e.g. [21].]

It is of interest to note that a similar property holds for Chern-Simons terms[3]: there is no generalization of the Chern-Simons terms involving higher order derivatives of the potentials that cannot be reexpressed as a local function of the field strengths and their derivatives [16, 20, 22, 23]. Adding derivatives does not lead therefore to new structure in the Chern-Simons context either.

Acknowledgements

This work is supported in part by the "Interuniversity Attraction Poles Programme – Belgian Science Policy ", by IISN-Belgium (convention 4.4505.86) and by the European Commission FP6 programme MRTN-CT-2004-005104, in which the authors are associated to the V.U.Brussel (Belgium).

A Appendix A: Covariant Poincaré lemma for linearized gravity

A.1 A useful preliminary lemma

Let us first recall a useful lemma proved in [24].

Lemma A.1 *Let η be a differential form depending only on the derivatives of a field (or set of fields) Φ and arbitrarily on a field (or set of fields) Ψ. Then*

$$\eta([\partial\Phi],[\Psi]) = d\omega([\Phi],[\Psi]) \Leftrightarrow \eta = d\Omega([\partial\Phi],[\Psi]) + \tilde{\eta}(d\Phi)\ , \quad \tilde{\eta}(0) = 0\,.$$

[3]We thank Stanley Deser for stressing this point to us.

In this equation, $\tilde{\eta}(d\Phi)$ is an exterior polynomial in the $d\Phi$'s, with coefficients that might involve the dx's (but we assume no explicit x-dependence).

A.2 Covariant Poincaré lemma

The covariant Poincaré lemma for linearized gravity has been stated first in the fundamental reference [24], where the BRST cohomology for full gravity was investigated (without antifields).

Covariant Poincaré Lemma:

Let η be a p-form. Then

$$d\eta([K_{\mu\nu\rho\sigma}]) = 0 \Leftrightarrow \eta = d\Omega([K_{\mu\nu\rho\sigma}]) + \tilde{\eta}(K^2_{\mu\nu})\,, \tag{A.1}$$

where $K^2_{\mu\nu} = \frac{1}{2}K_{\mu\nu\rho\sigma}dx^\rho dx^\sigma$ is the linearized curvature two-form. Again, $\tilde{\eta}(K^2_{\mu\nu})$ is here an exterior polynomial in the 2-forms $K^2_{\mu\nu}$.

This lemma has been proved in [24]. For completeness, we shall repeat it here, with one slight modification: namely we do not work in the vielbein formalism.

For $p = 0$, the lemma is trivial. Indeed $d\eta([K_{\mu\nu\rho\sigma}]) = 0$ implies that η is a constant by the usual Poincaré lemma. The lemma is also trivial when η does not involve the field h, so we consider that it is at least linear in h.

Let us proceed by induction. We assume that the lemma is true for all $p' < p$ and show that it is still valid for p. By the usual Poincaré lemma, $d\eta = 0$ implies that $\eta = d\omega_0([h])$. (The subscript denotes the ghost number.) Acting with γ_0 on $\eta = d\omega_0$ and using the usual Poincaré lemma, one obtains the following descent of equations (where $\omega_0 \equiv \hat{\omega}_0$)

$$\begin{aligned} \gamma_0\hat{\omega}_g &= d\hat{\omega}_{g+1} \quad 0 \leq g < G\,, \\ \gamma_0\hat{\omega}_G &= 0\,. \end{aligned}$$

The form degree of $\hat{\omega}_g$ is given by $p - g - 1$. The chain stops at some stage, either because some $\hat{\omega}_G$ is γ_0-closed, or because the form degree of $\hat{\omega}_G$ vanishes, i.e., $G = p - 1$.

Noting that $\gamma_0\omega_0$ contains the ghost C^ρ only with (at least) one derivative, Lemma A.1 together with $\gamma_0\omega_0 = d\hat{\omega}_1$ implies that $(\gamma_0\omega_0)([h], [\partial C]) = d\omega_1([h], [\partial C]) + \tilde{\omega}_1(dC)$, where $\tilde{\omega}_1(0) = 0$. One can iterate this step: $\gamma_0\omega_0 =$

$d\hat{\omega}_1$ and $\gamma_0\omega_0 = d\omega_1([h],[\partial C]) + \tilde{\omega}_1(dC)$ imply that $\omega_1([h],[\partial C]) = \hat{\omega}_1([h],[C]) - C^\sigma \frac{\partial \tilde{\omega}_1(dC)}{\partial(dC^\sigma)} + d\Omega_1([h],[C])$. Acting on the latter equation with γ_0, together with $\gamma_0\hat{\omega}_1 = d\hat{\omega}_2$, yields $\gamma_0\omega_1([h],[\partial C]) = d(\hat{\omega}_2([h],[C]) - \gamma_0\Omega_1([h],[C]))$. As γ_0 introduces only differentiated ghosts C, one can apply Lemma A.1 and conclude that $\gamma_0\omega_1 = d\omega_2([h],[\partial C]) + \tilde{\omega}_2(dC_\rho)$. The iteration leads to

$$\begin{aligned} \gamma_0\omega_g &= d\omega_{g+1} + \tilde{\omega}_{g+1}(dC) \quad 0 \leq g < G\,, \\ \gamma_0\omega_G &= \tilde{\omega}_{G+1}(dC)\,, \end{aligned} \tag{A.2}$$

where $\omega_g = \omega_g([h],[\partial C])$ for $0 \leq g \leq G$ and $\tilde{\omega}_g(0) = 0$.

Let us split the operator N as follows: $N = N_{[h]} + N_{[C]}$, where

$$N_{[h]} = \sum_s \partial_{\rho_1\cdots\rho_s} h_{\mu\nu} \frac{\partial}{\partial(\partial_{\rho_1\cdots\rho_s} h_{\mu\nu})}\,, \qquad N_{[C]} = \sum_s \partial_{\rho_1\cdots\rho_s} C^\mu \frac{\partial}{\partial(\partial_{\rho_1\cdots\rho_s} C^\mu)}\,.$$

We also define the operator $\tilde{N} = K + N$.

Equation $\eta = d\omega^0$ splits into eigenfunctions of N and $\tilde{N}$. It is sufficient to consider each eigenfunction separately. The following relations hold for all g (supposing the quantities involved to be nonvanishing):

$$N(\eta) = N(\omega_g) = N(\tilde{\omega}_g)\,,\ \tilde{N}(\eta) = \tilde{N}(\omega_g) + 1 = \tilde{N}(\tilde{\omega}_{g+1})\,.$$

As $\eta = d\omega_0$, $\tilde{N}(\eta)$ must be positive. Furthermore, since η depends on h only through the linearized curvature which contains two derivatives, $\tilde{N}(\eta) \geq 3N_{[h]}(\eta) = 3N(\eta)$. It is now straightforward to see that all $\tilde{\omega}$ vanish. Indeed, if $\tilde{\omega}_g(dC_\rho) \neq 0$, one would have

$$\tilde{N}(\eta) \geq 3N(\eta) = 3N(\tilde{\omega}_g) = \frac{3}{2}\tilde{N}(\tilde{\omega}_g) = \frac{3}{2}\tilde{N}(\eta)\,,$$

which is impossible for $\tilde{N}(\eta) > 0$.

The last equation of the descent becomes $\gamma_0\omega_G = 0$. This implies that

$$\omega_G = \bar{\omega}_G([K_{\mu\nu\rho\sigma}], H_{\mu\nu}) + \gamma_0\Lambda_{G-1}\,,$$

where we have taken into account that ω_G does not depend on the undifferentiated field C_ρ. (As a reminder, $H(\gamma_0)$ is generated by the linearized curvature and its derivatives, $[K_{\mu\nu\rho\sigma}]$, as well as by the ghosts C^ρ and $H^{\mu\nu}$, see e.g. [25]). For $G = 0$, this proves the lemma.

Let us consider $G > 0$. The second to last equation of (A.2) now reads $\gamma_0 \omega_{G-1} = d\bar{\omega}_G + d\gamma_0 \Lambda_{G-1}$. One splits the differential d into a part $\bar{d}$ that acts only on the fields h and a part $\tilde{d}$ that acts only on the ghosts. Then, $\tilde{d}\bar{\omega}_G$ is γ_0-exact and can be expressed as $\gamma_0 Y_{G-1}$. So the equation reads $\gamma_0(\omega_{G-1} - Y_{G-1} + d\Lambda_{G-1}) = \bar{d}\bar{\omega}_G$. Both sides have to vanish separately because the r.h.s. is a product of nontrivial elements of $H(\gamma_0)$ and cannot be γ_0-exact. Without loss of generality, one can write $\bar{\omega}_G = \sum_I P_I([K]) M^I_G(H^{\mu\nu})$, where P_I is a $(p - G - 1)$-form in the curvature and its derivatives, M^I is a polynomial in $H^{\mu\nu}$, and I denotes some set of indices. $\bar{d}\bar{\omega}_G = 0$ implies $dP_I([K]) = 0$. The induction hypothesis for $p' = p - G - 1$ can be used to solve this equation and the solution yields $P_I([K]) = \tilde{\eta}_I(K^2_{\mu\nu}) + dn_I([K])$, which implies $\bar{\omega}_G = \tilde{\eta}_I M^I_G + d(n_I M^I_G) - n_I d(M^I_G) = \tilde{\eta}_I M^I_G + dX_G + \gamma_0 \tilde{Y}_{G-1}$. One finally gets

$$\omega_G = \tilde{\eta}_I(K^2_{\mu\nu}) M^I_G(H^{\mu\nu}) \,,$$

up to trivial terms that can be removed by redefinitions of ω_G and ω_{G-1}.

We now claim that $G \leq 1$. Indeed, considering that $N_{[C]}(\omega_G) = G$, $K(\omega_G) = (2N_{[h]} + N_{[C]})(\omega_G)$ and $N_1 - N = K$, one has

$$\begin{aligned} K(\omega_0) + 1 = K(\eta) \geq 2N_{[h]}(\eta) = 2N(\eta) = 2N(\omega_G) &= (2N_{[h]} + 2N_{[C]})(\omega_G) \\ &= K(\omega_G) + G = K(\omega_0) + G \,, \end{aligned}$$

which implies $G \leq 1$. Accordingly, in order to complete the proof, we just need to treat the case $G = 1$, as the case $G = 0$ has already been solved.

From $\omega_1 = \frac{1}{2} H_{\mu\nu} \tilde{\eta}^{\mu\nu}(K^2)$ follows that $d\omega_1 = \gamma_0(-dx^\rho \partial_\mu h_{\nu\rho} \tilde{\eta}^{\mu\nu}(K^2))$. The most general solution ω_0 to $\gamma_0 \omega_0 = d\omega_1$ is $\omega_0 = -dx^\rho \partial_\mu h_{\nu\rho} \tilde{\eta}^{\mu\nu}(K^2) + \Omega([K_{\mu\nu\rho\sigma}])$. This in turn gives

$$\eta = d\omega_0 = K^2_{\mu\nu} \tilde{\eta}^{\mu\nu}(K^2) + d\Omega([K_{\mu\nu\rho\sigma}]) = \eta'(K^2) + d\Omega([K_{\mu\nu|\rho\sigma}]) \,,$$

where $\eta'(0) = 0$. This completes the proof of the lemma.

B Appendix B: Some results on $H(\gamma_0|d)$

In this appendix, we compute the conditions under which an element u of $H(\gamma_0)$ at the bottom of a descent of $H(\gamma_0|d)$ can be lifted once, i.e., satisfies the equation $du = \gamma_0 v$.

B.1 The differential D

Following [25], let us introduce the operator D such that

$$\begin{aligned} D\partial_{\nu_1}\ldots\partial_{\nu_s}h_{\mu\rho} &= d\partial_{\nu_1}\ldots\partial_{\nu_s}h_{\mu\rho}\,, \quad for\ all\ s \\ DC_\rho &= \frac{1}{2}dx^\nu H_{\nu\rho} \\ D\partial_{\nu_1}\ldots\partial_{\nu_s}C_\rho &= 0\,, \quad for\ s>0 \end{aligned}$$

The operator D is a differential. It is equal to the differential d up to γ_0-exact terms. It can be decomposed into a part D_0 acting on $h_{\mu\nu}$ and its derivatives, and a part D_1 acting on the C_μ and its derivatives. Let $\{\omega^I\}$ be a basis of the (finite-dimensional) space of the polynomials in C^ρ and $H^{\mu\nu}$. One has clearly $D_1\omega^I = A^I{}_J\omega^J$ for some matrices $A^I{}_J$. A grading is associated with the differential D, the D-grading, counting the number of $H^{\nu\rho}$. The operator D_0 leaves this grading invariant while D_1 raises it by one. It is useful to note that there is a maximal D-degree m for the ω^I, due to the fact that the H's anticommute.

B.2 "Liftable" cocycles

Let us now compute the γ_0-cocycles u that satisfy $du = \gamma_0 v$ for some v. We will assume that the form degree of u is smaller than n, as top-forms trivially satisfy the equation. Without loss of generality, one can also assume that u has the form $u = P_I\,\omega^I$, where $P_I = P_I([K_{\alpha\beta\gamma\delta}])$ is a polynomial in the linearized curvature and its derivatives up to some finite order (with coefficients that can involve the dx^μ) [25]. Then

$$\begin{aligned} du &= (dP_I)\omega^I + (-)^{|P_I|}P_I d\omega^I \\ &= (dP_I)\omega^I + (-)^{|P_I|}P_I D_1\omega^I + \gamma_0(\ldots) \\ &= (dP_J + (-)^{|P_I|}P_I A^I{}_J)\omega^J + \gamma_0(\ldots) \end{aligned}$$

where $|P_I|$ denotes the grassmanian parity of P_I. If this expression is to be γ_0-exact, then $(dP_J + (-)^{|P_I|}P_I A^I{}_J)\,\omega^J$ must vanish, or equivalently,

$$dP_J + (-)^{|P_I|}P_I A^I{}_J = 0\,. \tag{B.1}$$

We will expand Eq. (B.1) according to the D-degree. Let us note that it stops at the finite D-degree m. We denote by ω_i^J the basis elements of D-degree i, and by P_J^i their coefficient in u, so that $u = \sum_i P_J^i\omega_i^J$.

We will prove by induction that all P^i_J, $0 \leq i \leq m$, have the form

$$P^i_J = (-)^{|\Lambda^{i-1}|} \Lambda^{i-1}_I A^I_{i-1\,J} + d\Lambda^i_J + \tilde{P}^i_J \,, \tag{B.2}$$

where $\Lambda^i_J = \Lambda^i_J([K], dx)$ is at least linear in the curvature (and $\Lambda^{-1}_J = 0$), and $\tilde{P}^i_J = \tilde{P}^i_J(K^2_{\mu\nu}, dx)$ with $i < m$ is an exterior polynomial in the curvature 2-form that satisfies

$$\tilde{P}^i_J A^J_{i\,I} \omega^I_{i+1} = 0 \,. \tag{B.3}$$

In D-degree 0, Eq. (B.1) reads $dP^0_J = 0$, which implies, by the covariant Poincaré lemma (see Appendix A), $P^0_J = d\Lambda^0_J([K], dx) + \tilde{P}^0_J(K^2_{\mu\nu}, dx)$, for some Λ^0_J at least linear in the curvature. So P^0_J satisfies (B.2).

We now prove that if P^i_J, with $0 \leq i \leq m-1$, satisfies (B.2), then the equation in D-degree $i+1$ implies (B.3) for $\tilde{P}^i_J$ and (B.2) for P^{i+1}_J. Indeed, in D-degree $i+1$, the equation (B.1) reads

$$dP^{i+1}_J - (-)^{|\Lambda^i|} d\Lambda^i_I A^I_{i\,J} + (-)^{|\tilde{P}^i|} \tilde{P}^i{}_I A^I_{i\,J} = 0 \,.$$

The first two terms contain at least one differentiated curvature while the last term contains none. So the last term must vanish separately, which is exactly (B.3) for $\tilde{P}^i_J$. One is left with $d(P^{i+1}_J - (-)^{|\Lambda^i|} \Lambda^i_I A^I_{i\,J}) = 0$, which implies (B.2) for P^{i+1}_J by the covariant Poincaré lemma.

Inserting the expressions (B.2) for P^i_J into $u = P^i_J \omega^J_i$ yields (up to γ_0-exact terms)

$$u = \tilde{P}_J \omega^J + d(\Lambda_J)\omega^J + (-)^{|\Lambda_J|} \Lambda_J D_1 \omega^J = \tilde{P}_J(K^2_{\mu\nu}, dx)\omega^J + d(\Lambda_J([K], dx)\omega^J) \,.$$

To summarize:

In form degree $< n$, the γ_0-cocycles u that can be lifted at least once are given by

$$\tilde{P}_J(K^2_{\mu\nu}, dx)\omega^J(C^\rho, H^{\mu\nu})$$

where the exterior polynomials $\tilde{P}_J$ fulfill the constraint $\tilde{P}_J D_1 \omega^J = 0$ (where there is no sum over J), up to trivial γ_0- and d-exact terms.

B.3 An application

The constraint is automatically satisfied if u involves only the differentiated ghosts $H_{\alpha\beta}$ since $D_1 H_{\alpha\beta} = 0$. In form degree $n-2$ and ghost number one (a case needed in the text), a liftable u involving only $H_{\alpha\beta}$ reads thus

$$u = A^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_s\nu_s}_{\lambda_1\lambda_2\cdots\lambda_f\ \ \alpha\beta}\, K^2_{\mu_1\nu_1}\, K^2_{\mu_2\nu_2} \cdots K^2_{\mu_s\nu_s}\, dx^{\lambda_1} \cdots dx^{\lambda_f}\, H^{\alpha\beta} \tag{B.4}$$

with $n-2 = 2s+f$. The constant coefficients $A^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_s\nu_s}_{\lambda_1\lambda_2\cdots\lambda_f\ \ \alpha\beta}$ are antisymmetric under the exchange of μ_i with ν_i, the exchange of λ_i with λ_j and the exchange of α with β, and symmetric under the exchange of the pair (μ_i,ν_i) with the pair (μ_j,ν_j). In terms of the dual antisymmetric tensor $V^{\rho\sigma}$, this expression reads

$$V^{\rho\sigma} = B^{\rho\sigma\gamma_1\delta_1\gamma_2\delta_2\cdots\gamma_s\delta_s}_{\alpha\beta\mu_1\nu_1\mu_2\nu_2\cdots\mu_s\nu_s} K^{\mu_1\nu_1}{}_{\gamma_1\delta_1} K^{\mu_2\nu_2}{}_{\gamma_2\delta_2}\cdots K^{\mu_s\nu_s}{}_{\gamma_s\delta_s} H^{\alpha\beta} \tag{B.5}$$

where the constants $B^{\rho\sigma\gamma_1\delta_1\gamma_2\delta_2\cdots\gamma_s\delta_s}_{\alpha\beta\mu_1\nu_1\mu_2\nu_2\cdots\mu_s\nu_s}$ ($\sim \varepsilon A$) are completely antisymmetric in their upper indices,

$$B^{\rho\sigma\gamma_1\delta_1\gamma_2\delta_2\cdots\gamma_s\delta_s}_{\alpha\beta\mu_1\nu_1\mu_2\nu_2\cdots\mu_s\nu_s} = B^{[\rho\sigma\gamma_1\delta_1\gamma_2\delta_2\cdots\gamma_s\delta_s]}_{\alpha\beta\mu_1\nu_1\mu_2\nu_2\cdots\mu_s\nu_s},$$

and are antisymmetric under the exchange of α with β, and symmetric under the exchange of the pair (μ_i,ν_i) with the pair (μ_j,ν_j).

References

[1] C. Lanczos, Ann. Math. **39**, 842 (1938).

[2] D. Lovelock, Aequationes Math. **4**, 127 (1970).

[3] D. Lovelock, J. Math. Phys. **12**, 498 (1971).

[4] B. Zwiebach, Phys. Lett. B **156**, 315 (1985).

[5] B. Zumino, Phys. Rept. **137**, 109 (1986).

[6] C. Teitelboim and J. Zanelli, "Dimensionally Continued Topological Gravitation Theory in Hamltonian Form", Class. Quant. Grav. **4**,, L125 (1987);
M. Banados, C. Teitelboim and J. Zanelli, CECS-PHYS-1/90.

[7] R. C. Myers and J. Z. Simon, Phys. Rev. D **38**, 2434 (1988);
D. L. Wiltshire, Phys. Rev. D **38**, 2445 (1988);
T. Jacobson and R. C. Myers, Phys. Rev. Lett. **70**, 3684 (1993) [arXiv:hep-th/9305016];
J. Louko, J. Z. Simon and S. N. Winters-Hilt, Phys. Rev. D **55**, 3525 (1997) [arXiv:gr-qc/9610071].

[8] J. Crisostomo, R. Troncoso and J. Zanelli, Phys. Rev. D **62**, 084013 (2000) [arXiv:hep-th/0003271].

[9] M. Aiello, R. Ferraro and G. Giribet, Phys. Rev. D **70**, 104014 (2004) [arXiv:gr-qc/0408078];
M. Aiello, R. Ferraro and G. Giribet, arXiv:gr-qc/0502069.

[10] M. B. Green and J. H. Schwarz, Nucl. Phys. B **198** (1982) 441.

[11] A. A. Tseytlin, Nucl. Phys. B **584** (2000) 233 [arXiv:hep-th/0005072].

[12] K. Peeters, P. Vanhove and A. Westerberg, Class. Quant. Grav. **18** (2001) 843 [arXiv:hep-th/0010167].

[13] S. Deser and D. Seminara, Phys. Rev. D **62** (2000) 084010 [arXiv:hep-th/0002241].

[14] T. Damour and H. Nicolai, arXiv:hep-th/0504153.

[15] P.J. Olver, "Applications of Lie Groups to Differential Equations", Springer, New York: 1986.

[16] F. Brandt, N. Dragon and M. Kreuzer, Nucl. Phys. B **332**, 224 (1990).

[17] M. Dubois-Violette, M. Henneaux, M. Talon and C. M. Viallet, Phys. Lett. B **267**, 81 (1991).

[18] J. M. L. Fisch, M. Henneaux, J. Stasheff and C. Teitelboim, Commun. Math. Phys. **120**, 379 (1989),
J. M. L. Fisch and M. Henneaux, Commun. Math. Phys. **128**, 627 (1990);
M. Henneaux, Nucl. Phys. Proc. Suppl. **18A**, 47 (1990);
M. Henneaux, Nucl. Phys. Proc. Suppl. **57**, 131 (1997).

[19] M. Henneaux and C. Teitelboim, "Quantization of gauge systems," Princeton University Press (Princeton, New Jersey 1992).

[20] M. Dubois-Violette, M. Henneaux, M. Talon and C. M. Viallet, Phys. Lett. B **289**, 361 (1992) [arXiv:hep-th/9206106].

[21] N. Boulanger and S. Cnockaert, JHEP **0403**, 031 (2004) [arXiv:hep-th/0402180];
X. Bekaert, N. Boulanger and S. Cnockaert, J. Math. Phys. **46**, 012303 (2005) [arXiv:hep-th/0407102];
C. Bizdadea, C. C. Ciobirca, E. M. Cioroianu, S. O. Saliu and S. C. Sararu, Eur. Phys. J. C **36** (2004) 253 [arXiv:hep-th/0306154].

[22] G. Barnich, F. Brandt and M. Henneaux, Commun. Math. Phys. **174**, 93 (1995) [arXiv:hep-th/9405194].

[23] S. Deser and R. Jackiw, Phys. Lett. B **451**, 73 (1999) [arXiv:hep-th/9901125].

[24] F. Brandt, N. Dragon and M. Kreuzer, Nucl. Phys. B **340** (1990) 187.

[25] N. Boulanger, T. Damour, L. Gualtieri and M. Henneaux, Nucl. Phys. B **597**, 127 (2001) [arXiv:hep-th/0007220].